Chase Joyner

801 Homework 2

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Problem 1:

Let W be an $r \times s$ random matrix, and let A and C be $n \times r$ and $n \times s$ matrices of constants, respectively. Show that $\text{E}(AW + C) = A\text{E}(W) + C$. If B is an $s \times t$ matrix of constants, show that $\text{E}(AWB) = A\text{E}(W)B$. If $s = 1$, show that $\text{Cov}(AW + C) = A\text{Cov}(W)A'$.

Solution: Notice that the ijth element of the matrix $AW + C$ is

$$(AW + C)_{ij} = \sum_{k=1}^{s} a_{ik} w_{kj} + c_{ij}.$$

By linearity of expectations in one-dimension, we have

$$\text{E}\left(\sum_{k=1}^{s} a_{ik} w_{kj} + c_{ij}\right) = \sum_{k=1}^{s} a_{ik} \text{E}(w_{kj}) + c_{ij}.$$

By definition of $\text{E}(AW + C)$, we apply expectation to each element of this matrix. Therefore, this proves that

$$\text{E}(AW + C) = A\text{E}(W) + C.$$

Note by the first part of this question, we have $\text{E}(AWB) = A\text{E}(WB)$. All we need to show is that $\text{E}(WB) = \text{E}(W)B$. The ijth element of the matrix WB is

$$(WB)_{ij} = \sum_{k=1}^{s} w_{ik} b_{kj}.$$

Again, by linearity of expectations in one-dimension, we have

$$\text{E}\left(\sum_{k=1}^{s} w_{ik} b_{kj}\right) = \sum_{k=1}^{s} \text{E}(w_{ik}) b_{kj}.$$

Therefore, by definition of $\text{E}(WB)$, we take expectations component wise and so $\text{E}(WB) = \text{E}(W)B$. Thus, $\text{E}(AWB) = A\text{E}(W)B$. Lastly, we show that if $s = 1$, then $\text{Cov}(AW + C) =$

$A\text{Cov}(W)A'$. By definition of Covariances, we have

$$\begin{aligned}\text{Cov}(AW+C) &= \text{E}[(AW+C)(AW+C)'] - \text{E}[AW+C]\text{E}[AW+C]' \\ &= \text{E}[(AW)(AW)' + (AW)C' + C(AW)' + CC'] \\ &\quad - (A\text{E}(W))(A\text{E}(W))' + A\text{E}(W)C' + C(A\text{E}(W))' + CC' \\ &= A\text{E}[WW']A + A\text{E}(W)C' + C\text{E}(W')A' + CC' \\ &\quad - A\text{E}(W)\text{E}(W)'A' - A\text{E}(W)C' - C\text{E}(W)A' - CC' \\ &= A\text{E}(WW')A' - A\text{E}(W)\text{E}(W)'A' \\ &= A(\text{E}(WW') - \text{E}(W)\text{E}(W)')A' \\ &= A\text{Cov}(W)A'.\end{aligned}$$

Thus, we have proved the desired results.

Problem 2:

Show that $\text{Cov}(Y)$ is nonnegative definite for any random vector Y.

Solution: Assume that $Y \in \mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then, by problem 1, we have

$$x'\text{Cov}(Y)x = \text{Cov}(x'Y) = \text{Var}(x'Y) \geq 0.$$

Therefore, $\text{Cov}(Y)$ is nonnegative definite.

Problem 3:

Show that if Y is an r-dimensional random vector with $Y \sim N(\mu, V)$ and if B is a fixed $n \times r$ matrix, then $BY \sim N(B\mu, BVB')$.

Solution: Let Y be an r-dimensional random vector with $Y \sim N(\mu, V)$. Since V is a symmetric matrix, we can decompose it as $V = AA'$ for some vector A. Then, we observe that

$$Y \overset{d}{=} AZ + \mu.$$

Now let B be a fixed $n \times r$ matrix and so we have

$$BY \overset{d}{=} BAZ + B\mu.$$

Noticing that $(BA)(BA)' = BAA'B' = BVB'$, we conclude $BY \sim N(B\mu, BVB')$.

Problem 4:

Let M be the o.p.m onto $C(X)$. Show that $(I - M)$ is the o.p.m onto $C(X)^\perp$. Find $\text{tr}(I - M)$ in terms of $r(X)$.

Solution: Let M be the o.p.m onto $C(X)$, i.e.

$$\begin{aligned} x \in C(X) \quad &\Longrightarrow \quad Mx = x \\ y \in C(X)^\perp \quad &\Longrightarrow \quad My = 0. \end{aligned}$$

Then for any $x \in C(X)$,

$$Mx = x \quad \Longrightarrow \quad Mx - x = 0 \quad \Longrightarrow \quad (M - I)x = 0 \quad \Longrightarrow \quad (I - M)x = 0.$$

For any $y \in C(X)^\perp$,

$$My = 0 \quad \Longrightarrow \quad My - y = -y \quad \Longrightarrow \quad (M - I)y = -y \quad \Longrightarrow \quad (I - M)y = y.$$

Therefore, we conclude that $(I - M)$ is the o.p.m onto $C(X)^\perp$. Now we find $\text{tr}(I - M)$ in terms of $r(X)$. Let $M = OO'$, where $O = \left[o_1, ..., o_r\right]$ and $o_1, ..., o_r$ is an orthonormal basis for $C(X)$. By thm B.35, M is the o.p.m onto $C(X)$. Then, we have

$$\begin{aligned} \text{tr}(I - M) &= \text{tr}(I) - \text{tr}(M) = \text{tr}(I) - \text{tr}(OO') \\ &= n - r(OO') = n - r(M) = n - r(X). \end{aligned}$$

Therefore, we have $\text{tr}(I - M) = n - r(X)$.

Problem 5:

For a linear model $Y = X\beta + e$, $\text{E}(e) = 0$, $\text{Cov}(e) = \sigma^2 I$, show that $\text{E}(Y) = X\beta$ and $\text{Cov}(Y) = \sigma^2 I$.

Solution: By properties of expectations and covariance, we have

$$\text{E}(Y) = \text{E}(X\beta + e) = X\beta + \text{E}(e) = X\beta$$

and

$$\text{C}ov(Y) = \text{C}ov(X\beta + e) = \text{C}ov(e) = \sigma^2 I.$$

This shows the desired equalities.

Problem 6:

Let $Y = (y_1, y_2, y_3)'$ with $Y \sim N(\mu, V)$, where

$$\mu = (5, 6, 7)'$$

and

$$V = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

Find

(a) the marginal distribution of y_1,

(b) the joint distribution of y_1 and y_2.

(c) the conditional distribution of y_3 given $y_1 = u_1$ and $y_2 = u_2$,

(d) the conditional distribution of y_3 given $y_1 = u_1$,

(e) the conditional distribution of y_1 and y_2 given $y_3 = u_3$,

(f) the correlations $\rho_{12}, \rho_{13}, \rho_{23}$,

(g) the distribution of

$$Z = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} Y + \begin{bmatrix} -15 \\ -18 \end{bmatrix},$$

(h) the characteristic functions of Y and Z.

Solution:

(a) Define the row vector $B = (1, 0, 0)$. Then, by problem 3, we have

$$BY = y_1 \sim N(B\mu = 5, BVB' = 2).$$

Therefore, the marginal distribution of y_1 is $N(5, 2)$.

(b) Define the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, again by problem 3, we have

$$BY = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right).$$

(c) Let Y_1 and Y_2 be the vectors

$$Y_1 = \begin{bmatrix} y_3 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then we have that Y_1 given $Y_2 = \underset{\sim}{y}_2$ follows a Normal distribution with

$$\mu^\star = 7 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right) = 7 + \frac{1}{2}(u_1 - 5) + \frac{2}{3}(u_2 - 6)$$

and

$$V^\star = 4 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{13}{6}.$$

Therefore, $y_3 \mid y_1 = u_1, y_2 = u_2 \sim N\big(7 + \frac{1}{2}(u_1 - 5) + \frac{2}{3}(u_2 - 6), 13/6\big)$.

(d) First, we must obtain the joint distribution of y_1 and y_3. Define the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, by problem 3,

$$BY = \begin{bmatrix} y_1 \\ y_3 \end{bmatrix} \sim N\left(\begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \right).$$

Let $Y_1 = \begin{bmatrix} y_3 \end{bmatrix}$ and $Y_2 = \begin{bmatrix} y_1 \end{bmatrix}$. Then, Y_1 given $Y_2 = u_1$ follows a Normal distribution with

$$\mu^\star = 7 + 1(2)^{-1}(u_1 - 5) = 7 + \frac{1}{2}(u_1 - 5)$$

and

$$V^\star = 4 - 1(2)^{-1}1 = \frac{7}{2}.$$

Therefore, $y_3 \mid y_1 = u_1 \sim N\big(7 + \frac{1}{2}(u_1 - 5), \frac{7}{2}\big)$.

(e) Let Y_1 and Y_2 be the vectors

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} y_3 \end{bmatrix}.$$

Then, we have that Y_1 given $Y_2 = u_3$ follows a Normal distribution with

$$\mu^\star = \begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (4)^{-1}(u_3 - 7) = \begin{bmatrix} \frac{1}{4}u_3 + \frac{13}{4} \\ \frac{1}{2}u_3 + \frac{5}{2} \end{bmatrix}$$

and

$$V^\star = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} (4)^{-1} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 7/4 & -1/2 \\ -1/2 & 2 \end{bmatrix}.$$

Therefore, $y_1, y_2 \mid y_3 = u_3 \sim N(\mu^\star, V^\star)$.

(f) Recall the formula for correlation

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}.$$

Therefore, we see

$$\begin{aligned}
\rho_{12} &= \frac{v_{12}}{\sqrt{v_{11}v_{22}}} = \frac{0}{\sqrt{2 \cdot 3}} = 0 \\
\rho_{13} &= \frac{v_{13}}{\sqrt{v_{11}v_{33}}} = \frac{1}{\sqrt{2 \cdot 4}} = \frac{1}{\sqrt{8}} \\
\rho_{23} &= \frac{v_{23}}{\sqrt{v_{22}v_{33}}} = \frac{2}{\sqrt{3 \cdot 4}} = \frac{2}{\sqrt{12}}.
\end{aligned}$$

(g) Since $Y \sim N(\mu, V)$, then

$$Y \overset{d}{=} AZ + \mu.$$

Therefore, we have

$$BY + \mu^\star \overset{d}{=} BAZ + B\mu + \mu^\star.$$

This implies that $BY + \mu^\star \sim N(B\mu + \mu^\star, BVB')$, where

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mu^\star = \begin{bmatrix} -15 \\ -18 \end{bmatrix}.$$

Therefore, we have the distribution of Z to be

$$Z \sim N\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 & 11 \\ 11 & 15 \end{bmatrix} \right).$$

(h) Recall that the characteristic function for the multvariate normal distribution is

$$\Phi(t) = \exp\left\{ it'\mu - \frac{1}{2} t'Vt \right\}.$$

Let $t = (t_1, t_2, t_3)'$. Then, we have that

$$\Phi_Y(t) = \exp\left\{ it' \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} - \frac{1}{2} t' \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} t \right\}$$

and

$$\Phi_Z(t) = \exp\left\{ it' \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} t' \begin{bmatrix} 11 & 11 \\ 11 & 15 \end{bmatrix} t \right\}.$$

Problem 7:

The density of $Y = (y_1, y_2, y_3)'$ is

$$(2\pi)^{-3/2}|V|^{-1/2}e^{-Q/2},$$

where

$$Q = 2y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 - 8y_1 - 4y_2 + 8.$$

Find V^{-1} and μ.

Solution: This multivariate normal distribution can be written as

$$(2\pi)^{-3/2}|V|^{-1/2}\exp\left\{-\frac{1}{2}(Y-\mu)'V^{-1}(Y-\mu)\right\}.$$

This implies that

$$\begin{aligned} Q &= (Y-\mu)'V^{-1}(Y-\mu) \\ &= Y'V^{-1}Y - 2\mu'V^{-1}Y + \mu'V^{-1}\mu \\ &= 2y_1^2 + y_2^2 + y_3^2 + 2y_1y_2 - 8y_1 - 4y_2 + 8. \end{aligned}$$

Solving the above equality for V^{-1} and μ, we find

$$V^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 8:

Let $Y = (y_1, y_2, y_3)' \sim N(\mu, \sigma^2 I)$. Consider the quadratic forms defined by the matrices M_1, M_2, and M_3 given below.

$$M_1 = \frac{1}{3}J_3^3, \quad M_2 = \frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}, \quad M_3 = \frac{1}{42}\begin{bmatrix} 1 & -5 & 4 \\ -5 & 25 & -20 \\ 4 & -20 & 16 \end{bmatrix}.$$

(a) Find the distribution of each $Y'M_iY$.

(b) Show that the quadratic forms are pairwise independent.

(c) Show that the quadratic forms are mutually independent.

Solution:

(a) First we must show that M_i is an o.p.m for $i = 1, 2, 3$; i.e. $M_iM_i = M_i$ and $M_i' = M_i$. It is easily seen that $M_i' = M_i$ for $i = 1, 2, 3$. Also, squaring each matrix will show that M_i is idempotent for $i = 1, 2, 3$. Therefore, M_i is an o.p.m for $i = 1, 2, 3$. Now since $Y \sim N(\mu, \sigma^2 I)$ and M_i is idempotent, we have

$$Y'M_iY/\sigma^2 \sim \chi^2\big(tr(M_i), \mu'M_i\mu/(2\sigma^2)\big) = \chi^2\big(1, \mu'M_i\mu/(2\sigma^2)\big)$$

for $i = 1, 2, 3$.

(b) Recall that if $Y \sim N(\mu, \sigma^2 I)$ and $BA = 0$, then $Y'AY$ and $Y'BY$ are independent. Therefore, we just need to show that $M_i M_j = 0$ for $i \neq j$ to establish pairwise independence. So, we have

$$M_1 M_2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^3 \frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_1 M_3 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^3 \frac{1}{42}\begin{bmatrix} 1 & -5 & 4 \\ -5 & 25 & -20 \\ 4 & -20 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_2 M_3 = \frac{1}{14}\begin{bmatrix} 9 & -3 & -6 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix} \frac{1}{42}\begin{bmatrix} 1 & -5 & 4 \\ -5 & 25 & -20 \\ 4 & -20 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, we conclude that $Y'M_iY$ are pairwise independent.

(c) To establish mutual independence, we will first show that the M_iY's are mutually independent. To do this, note by problem 3 we have the distribution

$$\begin{aligned} \begin{bmatrix} M_1Y \\ M_2Y \\ M_3Y \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} Y &\sim N\left(\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \mu, \sigma^2 \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} I \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}' \right) \\ &= N\left(\begin{bmatrix} M_1\mu \\ M_2\mu \\ M_3\mu \end{bmatrix}, \sigma^2 \begin{bmatrix} M_1M_1' & M_1M_2' & M_1M_3' \\ M_2M_1' & M_2M_2' & M_2M_3' \\ M_3M_1' & M_3M_2' & M_3M_3' \end{bmatrix} \right). \end{aligned} \tag{1}$$

By part (b), we conclude the covariance matrix in distribution (1) becomes

$$\begin{bmatrix} M_1Y \\ M_2Y \\ M_3Y \end{bmatrix} \sim N\left(\begin{bmatrix} M_1\mu \\ M_2\mu \\ M_3\mu \end{bmatrix}, \sigma^2 \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \right).$$

Since the off diagonals of the covariance matrix in this joint distribution are all 0, the M_iY's are mutually independent. Now consider the function of M_iY, namely $(M_iY)'(M_iY) = Y'M_iY$. Since the M_iY's are mutually independent, any function of them should be as well. Thus, the $Y'M_iY$'s are mutually independent.